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THE EVOLUTION OF THE MOTION OF A VISCOELASTIC SPHERE IN THE RESTRICTED CIRCULAR THREE-BODY PROBLEM†

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The restricted circular three-body problem is investigated when two of the massive bodies, which are treated as point masses, move in specified circular orbits in a single plane while the third body of small mass is assumed to be spherically symmetric and deformable and its centre of mass moves in the plane of the circular orbits of the first two bodies and rotation around the centre of mass occurs around the normal to the plane of motion of the centre of mass. The energy dissipation accompanying the deformations of the small, spherically symmetric, deformable body is an important factor affecting the evolution of its motion. This energy dissipation leads to the evolution of its orbit and angular velocity of rotation. Since it is assumed that the masses of the two bodies (in the case of the solar system, these could be the Sun and Jupiter) relate as one to μ ($\mu \ll 1$), the evolution of the motion of the deformable body develops in two stages. During the first, "fast" stage of evolution, its orbit tends towards circular with its centre in the massive body with mass equal to unity, and the rotation is identical to the orbital rotation (a state of gravitational stabilization, 1:1 resonance). In this case, the body turns out to be deformed (oblate with respect to its poles and stretched along the radius which joins this body of small mass to the massive body [1, 2]. In the second, "slow" stage of evolution, the effect of the body with mass μ is taken into consideration, which leads to the evolution of the circular orbit of the deformable body. © 2001 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM. THE EQUATIONS OF MOTION

Suppose two point masses M_1 and M_2, the masses of which are equal to unity and μ ($\mu \ll 1$) move in circular orbits under the action of Newtonian gravitation forces around a common centre of mass O in the OXY plane. Suppose that $OM_2 = b$, $OM_1 = \mu b$, and that the angle α between the OX axis and the radius vector $\mathbf{OM}_2$ of the point M_2 changes according to the law

$$\alpha(t) = \frac{\omega_3 t}{1+\mu} + \alpha(0), \quad \omega_3 = \sqrt{\frac{f}{b^3}}$$

where f is the gravitational constant (Fig. 1).

Further, assume that the centre of mass C of a viscoelastic, deformable, homogeneous sphere of mass m and density ρ moves in the OXY plane and that $\mathbf{R}$ is the radius vector of the point C. The position of the points of the sphere is determined by the vector field

$$\zeta(\mathbf{r},t) = \mathbf{R}(t) + O(t)(\mathbf{r} + \mathbf{u}(\mathbf{r},t))$$
$$\mathbf{R}(t) = \frac{1}{m}\int_V \zeta(\mathbf{r},t)\rho dx, \quad \int_V \mathbf{u} dx = \int_V \operatorname{rot} \mathbf{u} dx = 0, \quad dx = dx_1 dx_2 dx_3 \tag{1.1}$$

Here $V = \{\mathbf{r} : |\mathbf{r}| < r_0\}$ is the domain in E^3 which is occupied by the sphere in its natural undeformed state, conditions (1.1) uniquely define the radius vector $\mathbf{R}(t)$ of the centre of mass C of the deformable sphere and the system of coordinates $Cx_1x_2x_3$, relative to which the sphere, in an integral sense, does not rotate [1]. The operator $O(t) = O(\varphi(t))$ defines the transition from the system of coordinates $Cx_1x_2x_3$ to the König system of axes $C\xi_1\xi_2\xi_3$ and has the form

$$O(\varphi) = \begin{Vmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

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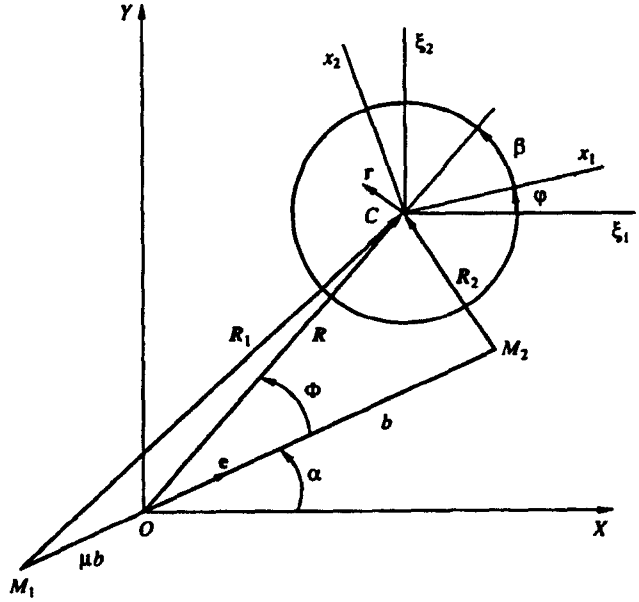

Fig. 1

The kinetic energy of the sphere is represented by the functional

$$T = \frac{1}{2}\int_V \dot{\zeta}^2 \rho dx = \frac{1}{2}\int_V [O^{-1}\dot{\mathbf{R}} + \boldsymbol{\omega}\times(\mathbf{r}+\mathbf{u}) + \dot{\mathbf{u}}]^2 \rho dx$$

where $\boldsymbol{\omega}$ is the angular velocity vector which is defined by the equality $\boldsymbol{\omega} \times (\cdot) = O^{-1}O(\cdot)$ and has the form

$$\boldsymbol{\omega} = \varphi \mathbf{e}_3, \quad \mathbf{e}_3 = (0, 0, 1)$$

In accordance with conditions (1.1), we obtain

$$T = \frac{1}{2}m\dot{\mathbf{R}}^2 + \frac{1}{2}\int_V \dot{\varphi}^2[\mathbf{e}_3 \times (\mathbf{r}+\mathbf{u})]^2 \rho dx + \int_V (\dot{\varphi}\mathbf{e}_3 \times (\mathbf{r}+\mathbf{u}), \dot{\mathbf{u}})\rho dx + \frac{1}{2}\int_V \dot{\mathbf{u}}^2 \rho dx$$

The potential energy functional has the form

$$\Pi = -\int_V \frac{f\rho dx}{\sqrt{(\mathbf{R}_1 + O(\mathbf{r}+\mathbf{u}))^2}} - \int_V \frac{\mu f \rho dx}{\sqrt{(\mathbf{R}_2 + O(\mathbf{r}+\mathbf{u}))^2}} + \mathscr{E}[\mathbf{u}] \tag{1.2}$$

$$\mathbf{R}_1 = \mathbf{M}_1\mathbf{C} = \mathbf{R} + \mu b\mathbf{e}, \quad \mathbf{R}_2 = \mathbf{M}_2\mathbf{C} = \mathbf{R} - b\mathbf{e}, \quad \mathbf{e} = (\cos\alpha, \sin\alpha, 0) \tag{1.3}$$

$$\mathscr{E}[\mathbf{u}] = \int_V a(I_E^2 - a_1' II_E)dx, \quad a > 0, \quad 0 < a_1' < 3$$

$$a = \frac{\mathrm{E}(1-\nu)}{2(1+\nu)(1-2\nu)}, \quad a_1' = \frac{2(1-2\nu)}{1-\nu}$$

$$I_E = \sum_{i=1}^{3} e_{ii}, \quad II_E = \sum_{i<j}^{3} (e_{ii}e_{jj} - e_{ij}^2), \quad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$\mathscr{E}[\mathbf{u}]$ is the functional of the potential energy of elastic deformations, corresponding to the classical theory of the elasticity at small deformations [1], E is Young's modulus of elasticity and ν is Poisson's ratio.

Since $|\mathbf{R}_k| \gg |\mathbf{r} + \mathbf{u}|$ $(k = 1, 2)$, the integrand in (1.2) can be expanded in series. Restricting the treatment to terms of the third order with respect to the powers of $|\mathbf{r} + \mathbf{u}|/R_k$ and linear with respect to $|\mathbf{u}|/R_k$, where $R_k = |\mathbf{R}_k|$ $(k = 1, 2)$, we obtain

$$\Pi = -\frac{fm}{R_1} - \frac{\mu fm}{R_2} + \frac{f}{R_1^3}\int_V \{(\mathbf{r},\mathbf{u}) - 3(O^{-1}\mathbf{R}_{10},\mathbf{r})(O^{-1}\mathbf{R}_{10},\mathbf{u})\}\rho dx +$$
$$+ \frac{\mu f}{R_2^3}\int_V \{(\mathbf{r},\mathbf{u}) - 3(O^{-1}\mathbf{R}_{20},\mathbf{r})(O^{-1}\mathbf{R}_{20},\mathbf{u})\}\rho dx + \mathscr{E}[\mathbf{u}]$$

Here

$$\mathbf{R}_{k0} = \mathbf{R}_k / R_k, \quad k = 1,2$$
$$R_1 = (R^2 + 2\mu Rb\cos\Phi + \mu^2 b^2)^{1/2}, \quad R_2 = (R^2 - 2Rb\cos\Phi + b^2)^{1/2}$$

$R = |\mathbf{R}|$, and Φ is the angle between the vectors $\mathbf{OC}$ and $\mathbf{OM}_2$.

We write the equations of motion in the form of Routh's equations, using the canonical Delaunay–Andoyer variables (I, L, G, φ, l, g) which define the motion of the centre of mass of the sphere and its rotation around that Cx_3 axis and the Lagrangian variables $u_1(\mathbf{r}, t)$, $u_2(\mathbf{r}, t)$, $u_3(\mathbf{r}, t)$.

The modulus of the angular momentum vector of the sphere with respect to the system of coordinates $C\xi_1\xi_2\xi_3$ is defined by the equality

$$I = \nabla_{\dot\varphi} T = J_{33}[\mathbf{u}]\dot\varphi + G_u \tag{1.4}$$
$$J_{33}[\mathbf{u}] = \int_V [\mathbf{e}_3 \times (\mathbf{r}+\mathbf{u})]^2 \rho dx, \quad G_u = (\mathbf{e}_3, \int_V [(\mathbf{r}+\mathbf{u})\times\dot{\mathbf{u}}]\rho dx)$$

The change from the variables $(\mathbf{R}, \dot{\mathbf{R}})$ to the canonical Delaunay variables (L, G, l, g) is achieved using the relations [3, 4]

$$\frac{1}{2}m\dot{\mathbf{R}}^2 - \frac{fm}{R} = -\frac{f^2m^3}{2L^2} \tag{1.5}$$
$$R = \frac{G^2}{fm^2(1+e\cos\vartheta)}, \quad \cos w = \frac{e+\cos\vartheta}{1+e\cos\vartheta}, \quad l = w - e\sin w, \quad e = \sqrt{1-\frac{G^2}{L^2}}$$

Here e is the eccentricity of the orbit of the centre of mass of the sphere ϑ, l, w are the actual, mean and eccentric anomalies respectively and g is the longitude of the perihelion from the ascending node. In the Delaunay variables

$$\mathbf{R} = (R\cos(g+\vartheta),\ R\sin(g+\vartheta), 0), \quad \Phi = g + \vartheta - \alpha$$

The Routh functional is defined by the equality

$$\mathfrak{R} = T_2 - T_0 + \Pi$$
$$T_2 = \frac{1}{2}m\dot{\mathbf{R}}^2 + \frac{1}{2}\int_V \dot\varphi^2[\mathbf{e}_3\times(\mathbf{r}+\mathbf{u})]^2\rho dx, \quad T_0 = \frac{1}{2}\int_V \dot{\mathbf{u}}^2\rho dx$$

By virtue of the transformations (1.4) and (1.5), we obtain

$$\mathfrak{R}[I, L, G, \varphi, l, g, \dot{\mathbf{u}}, \mathbf{u}, \alpha] = \frac{(I-G_u)^2}{2J_{33}[\mathbf{u}]} - \frac{f^2m^3}{2L^2} - \frac{1}{2}\int_V \dot{\mathbf{u}}^2\rho dx +$$
$$+ \mu fm\left(\frac{R_1 - R}{\mu R_1 R} - \frac{1}{R_2}\right) + \frac{f}{R_1^3}\int_V \{(\mathbf{r},\mathbf{u}) - 3(O^{-1}\mathbf{R}_{10},\mathbf{r})(O^{-1}\mathbf{R}_{10},\mathbf{u})\}\rho dx +$$
$$+ \frac{\mu f}{R_2^3}\int_V \{(\mathbf{r},\mathbf{u}) - 3(O^{-1}\mathbf{R}_{20},\mathbf{r})(O^{-1}\mathbf{R}_{20},\mathbf{u})\}\rho dx + \mathscr{E}[\mathbf{u}] \tag{1.6}$$

The functional J_{33} [$\mathbf{u}$] can be represented in the form of the sum of the moment of inertia of the undeformed sphere with respect to the $C\xi_3$ axis, the linear part of the functional J_{33} [$\mathbf{u}$] with respect to the components of the vector $\mathbf{u}$ and the quadratic part of this functional with respect to the components of the vector $\mathbf{u}$

$$J_{33}[\mathbf{u}] = A + J_{33}^{(1)}[\mathbf{u}] + J_{33}^{(2)}[\mathbf{u}]$$

$$A = \frac{2}{5}mr_0^2, \quad J_{33}^{(1)}[\mathbf{u}] = 2\int_V [(\mathbf{r},\mathbf{u}) - (\mathbf{e}_3,\mathbf{r})(\mathbf{e}_3,\mathbf{u})]\rho dx, \quad J_{33}^{(2)}[\mathbf{u}] = \int_V [\mathbf{u}^2 - (\mathbf{e}_3,\mathbf{u})^2]\rho dx$$

The equations of motion of the viscoelastic sphere are written in the form of Routh's equations

$$\dot{I} = -\frac{\partial\mathfrak{R}}{\partial\varphi}, \quad \dot{\varphi} = \frac{\partial\mathfrak{R}}{\partial I}, \quad \dot{L} = -\frac{\partial\mathfrak{R}}{\partial l}, \quad \dot{l} = \frac{\partial\mathfrak{R}}{\partial L}, \quad \dot{G} = -\frac{\partial\mathfrak{R}}{\partial g}, \quad \dot{g} = \frac{\partial\mathfrak{R}}{\partial G} \tag{1.7}$$

$$\int_V \left[\left(-\frac{d}{dt}\nabla_{\dot{\mathbf{u}}}\mathfrak{R} + \nabla_{\mathbf{u}}\mathfrak{R} + \nabla_{\dot{\mathbf{u}}}\mathfrak{D} + \boldsymbol{\lambda}_1\right)\delta\mathbf{u} + \boldsymbol{\lambda}_2 \operatorname{rot}\delta\mathbf{u}\right]dx = 0, \quad \forall\delta\mathbf{u} \in (W_2^1(V))^3 \tag{1.8}$$

where $(W_2^1(V))^3$ is the Sobolev space $\mathfrak{D}[\dot{\mathbf{u}}] = \chi\mathscr{E}[\dot{\mathbf{u}}]$ is a dissipative functional and χ is the coefficient of internal viscous friction. The last equation is written in the form of the d'Alembert–Lagrange variational principle and contains the two undetermined multipliers $\boldsymbol{\lambda}_1(t)$ and $\boldsymbol{\lambda}_2(t)$, which are generated by conditions (1.1).

We shall assume that lowest frequency of the natural oscillations of the sphere is far greater than the angular velocities $\dot{\Phi}$and ω_3. This means that, for an appropriate choice of the scales of the dimensioned quantities, the numerical value of the modulus of elasticity of the material of the sphere E will be large and the parameter $\varepsilon = \mathrm{E}^{-1}$ will be small. The displacements $\mathbf{u}(\mathbf{r}, t)$ after the decay of the natural oscillations due to the existence of dissipative forces will be proportional to the small parameter ε

$$\mathbf{u}(\mathbf{r}, t) = \varepsilon\mathbf{u}_1(\mathbf{r}, t)$$

If it is assumed that $\varepsilon \neq 0$, $\mu = 0$, the system of equations (1.7) and (1.8) will describe the evolution of the motion of a viscoelastic sphere in a central Newtonian force field, subject to the condition that the orbital plane of the centre of mass remains fixed and rotation around the centre of mass occurs around the normal to this plane. A spatial version of this problem, as well as the above-mentioned special case, have been studied in [1, 5]. It was shown that, when there is energy dissipation, the deformable planet tends to a steady motion in which the centre of mass of the planet describes a circle and its orientation is unchanged in the orbital axes. The rate of this evolution is of the order of $\varepsilon\chi$.

We will now study the case when $\varepsilon \neq 0$, $\mu \neq 0$ and determine how the existence of a third body (a planet of mass μ) affects the evolution of the motion of the viscoelastic sphere.

2. CONSTRUCTION OF THE APPROXIMATE EVOLUTIONARY EQUATIONS OF MOTION OF A VISCOELASTIC SPHERE IN THE RESTRICTED THREE-BODY PROBLEM

Taking account of formulae (1.3), we represent the Routh functional (1.6) in the form

$$\mathfrak{R} = \frac{(I - G_u)^2}{2J_{33}[\mathbf{u}]} - \frac{f^2m^3}{2L^2} - \frac{1}{2}\int_V \dot{\mathbf{u}}^2\rho dx + H[\mu, R, \Phi, \beta, \mathbf{u}] + \mathscr{E}[\mathbf{u}] \tag{2.1}$$

$$\Phi = g + \vartheta - \alpha, \quad \beta = g + \vartheta - \varphi$$

$$H[\mu, R, \Phi, \beta, \mathbf{u}] = F_0 + \int_V \{F_1(\mathbf{r},\mathbf{u}) - F_2(\xi_0,\mathbf{r})(\xi_0,\mathbf{u}) -$$

$$-F_3[(\xi_0,\mathbf{r})(\xi_1,\mathbf{u}) + (\xi_1,\mathbf{r})(\xi_0,\mathbf{u})] - F_4(\xi_1,\mathbf{r})(\xi_1,\mathbf{u})\}dx$$

$$\xi_0 = O^{-1}\mathbf{R}_0 = (\cos\beta, \sin\beta, 0)$$

$$\xi_1 = O^{-1}\mathbf{e} = (\cos(\beta - \Phi), \sin(\beta - \Phi), 0)$$

$$F_i = F_i(\mu, R, \Phi), \quad i = 0, 1, \ldots, 4$$

Hence, the dependence of the functional H on the variables L, G, l, g, φ in terms of the functions R, Φ, β is achieved using formulae (1.5) and (2.1).

The functions F_i ($i = 0, 1, \ldots, 4$) can be expanded in series in powers of the small parameter μ. Restricting the treatment to terms of zero- and first-order infinitesimals in μ, we obtain

$$\begin{aligned}
&F_0 = \mu f_{01}, \quad f_{01} = fmR^{-1}(q\cos\Phi - p)\\
&F_1 = f_{10} + \mu f_{11}, \quad f_{10} = \rho fR^{-3}, \quad f_{11} = \rho fR^{-3}(p^3 - 3q\cos\Phi)\\
&F_2 = f_{20} + \mu f_{21}, \quad f_{20} = 3\,\rho fR^{-3}, \quad f_{21} = 3\,\rho fR^{-3}\,(p^5 - 5q\cos\Phi)\\
&F_3 = \mu f_{31}, \quad f_{31} = 3\rho fR^{-3}\,(q - qp^5)\\
&F_4 = \mu f_{41}, \quad f_{41} = 3\rho fR^{-3}q^2p^5\\
&q = bR^{-1}, \quad p = (1 - 2q\cos\Phi + q^2)^{-1/2}
\end{aligned}$$

We will henceforth restrict the treatment to considering a class of quasicircular orbits, that is, orbits with an eccentricity equal to zero. For this purpose, we will show that Eqs (1.7) admit of the set of solutions indicated below.

Lemma 1. The system of equations (1.7) has, as its solutions, the class of quasicircular orbits when the eccentricity $e(t) = 0$.

Proof. Consider the difference $L - G$ we will show that, by virtue of Eqs (1.7), the time derivative of this difference vanishes when $e = 0$. In fact

$$\dot{L} - \dot{G} = \frac{\partial\mathfrak{R}}{\partial g} - \frac{\partial\mathfrak{R}}{\partial l} = \frac{\partial H}{\partial g} - \frac{\partial H}{\partial l} =$$

$$= \frac{\partial H}{\partial\Phi}\frac{\partial\Phi}{\partial g} + \frac{\partial H}{\partial\beta}\frac{\partial\beta}{\partial g} - \frac{\partial H}{\partial R}\frac{\partial R}{\partial l} - \frac{\partial H}{\partial\Phi}\frac{\partial\Phi}{\partial l} - \frac{\partial H}{\partial\beta}\frac{\partial\beta}{\partial l}$$

From the definition of the functions Φ and β, according to formulae (2.1), we obtain

$$\frac{\partial\Phi}{\partial g} = 1, \quad \frac{\partial\beta}{\partial g} = 1, \quad \frac{\partial\Phi}{\partial l} = \frac{\partial\Phi}{\partial\vartheta}\frac{\partial\vartheta}{\partial l} = \frac{\partial\vartheta}{\partial l}, \quad \frac{\partial\beta}{\partial l} = \frac{\partial\beta}{\partial\vartheta}\frac{\partial\vartheta}{\partial l} = \frac{\partial\vartheta}{\partial l}$$

Next, according to the transformation (1.5)

$$\frac{\partial R}{\partial l} = \frac{L^3}{fm^2G}e\sin\vartheta, \quad \frac{\partial\vartheta}{\partial l} = \frac{(1 + e\cos\vartheta)^2}{(1 - e^2)^{3/2}}$$

When $e = 0$, we have $\partial R/\partial l = 0$, $\partial\vartheta/\partial l = 1$. Consequently $\dot{L} - \dot{G} = 0$. Lemma 1 is proved.

Since, for the class of quasicircular orbits, the Delaunay variables degenerate, it is advisable in this case to use the canonical Poincaré variables [6]

$$\Lambda = L, \quad \Gamma = L - G, \quad \lambda = l + g, \quad \gamma = -g$$

For the class of quasicircular orbits ($e = 0$)

$$\Gamma(t) = 0, \quad R = \Lambda^2 / fm^2, \quad \vartheta = l, \quad \Phi = \lambda - \alpha, \quad \beta = \lambda - \varphi \tag{2.2}$$

The Routh functional in the Andoyer–Poincaré variables has the form

$$\mathfrak{R}[I, \Lambda, \varphi, \lambda, \dot{\mathbf{u}}, \mathbf{u}, \alpha] = \frac{(I - G_u)^2}{2J_{33}[\mathbf{u}]} - \frac{f^2m^3}{2\Lambda^2} - \frac{1}{2}\int_V \dot{\mathbf{u}}^2\rho dx +$$

$$+ H[\mu, R, \Phi, \beta, \mathbf{u}] + \mathscr{E}[\mathbf{u}]$$

and R, Φ, β are defined by relations (2.2).

We represent the equations of motion in the form

$$
\begin{aligned}
\dot{I} &= -\frac{\partial \mathfrak{R}}{\partial \varphi} = -\frac{\partial H}{\partial \varphi} = \frac{\partial H}{\partial \beta} \\
\dot{\Lambda} &= -\frac{\partial \mathfrak{R}}{\partial \lambda} = -\frac{\partial H}{\partial \lambda} = -\frac{\partial H}{\partial \Phi} - \frac{\partial H}{\partial \beta} \\
\dot{\beta} &= \frac{\partial \mathfrak{R}}{\partial \Lambda} - \frac{\partial \mathfrak{R}}{\partial I} = \frac{f^2 m^3}{\Lambda^3} - \frac{I - G_u}{J_{33}[\mathbf{u}]} + \frac{\partial H}{\partial \Lambda} \\
\dot{\Phi} &= \frac{\partial \mathfrak{R}}{\partial \Lambda} - \frac{\omega_3}{1+\mu} = \frac{f^2 m^3}{\Lambda^3} + \frac{\partial H}{\partial \Lambda} - \frac{\omega_3}{1+\mu}
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
&\int_V [(\rho \ddot{\mathbf{u}} + \frac{d}{dt}\left\{\rho \frac{I - G_u}{J_{33}[\mathbf{u}]}[\mathbf{e}_3 \times (\mathbf{r} + \mathbf{u})]\right\} + \rho \frac{I - G_u}{J_{33}[\mathbf{u}]}[\mathbf{e}_3 \times \dot{\mathbf{u}}] - \\
&- \rho \frac{(I - G_u)^2}{J_{33}^2[\mathbf{u}]}[\mathbf{r} + \mathbf{u} - (\mathbf{e}_3, \mathbf{r} + \mathbf{u})\mathbf{e}_3] + \nabla_{\mathbf{u}} H + \nabla_{\mathbf{u}} \mathscr{E} + \nabla_{\dot{\mathbf{u}}} \mathscr{D} + \boldsymbol{\lambda}_1)\delta \mathbf{u} + \\
&+ \boldsymbol{\lambda}_2 \operatorname{rot} \delta \mathbf{u}]dx = 0, \quad \forall \delta \mathbf{u} \in (W_2^1(V))^3
\end{aligned}
\tag{2.4}
$$

If $\varepsilon = 0$, then $\mathbf{u}(\mathbf{r}, t) = 0$ and Eqs (2.3) take the form

$$
\dot{I} = 0, \quad \dot{\Lambda} = -\frac{\partial F_0}{\partial \Phi}, \quad \dot{\beta} = \omega_2 - \omega_1 + \frac{\partial F_0}{\partial \Lambda}, \quad \dot{\Phi} = \omega_2 - \frac{\omega_3}{1+\mu} + \frac{\partial F_0}{\partial \Lambda}
\tag{2.5}
$$

Here

$$
\omega_1 = \frac{I}{A}, \quad \omega_2 = \frac{f^2 m^3}{\Lambda^3}, \quad \omega_3 = \left(\frac{f}{b^3}\right)^{1/2}
\tag{2.6}
$$

Equations (2.5) describe the motion of a sphere as a rigid body in the classical restricted three-body problem in the class of quasicircular orbits.

When $\varepsilon \neq 0$, according to the method of separation of the motions [1], after the decay of the natural oscillations of the viscoelastic sphere the solution $\mathbf{u}(\mathbf{r}, t)$ is sought in the form

$$
\mathbf{u}(\mathbf{r}, t) = \varepsilon \mathbf{u}_1(\mathbf{r}, t) + \ldots
$$

Here, it is also necessary to seek the Lagrange multipliers $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ in the form of expansions in powers of ε

$$
\begin{aligned}
\boldsymbol{\lambda}_1(t) &= \boldsymbol{\lambda}_{10}(t) + \varepsilon \boldsymbol{\lambda}_{11}(t) + \ldots \\
\boldsymbol{\lambda}_2(t) &= \boldsymbol{\lambda}_{20}(t) + \varepsilon \boldsymbol{\lambda}_{21}(t) + \ldots
\end{aligned}
$$

The equation for the function $\mathbf{u}_1(\mathbf{r}, t)$ takes the form

$$
\begin{aligned}
&\int_V \left[\frac{d}{dt}\left(\rho \frac{I}{A}\mathbf{e}_3 \times \mathbf{r}\right) - \rho \frac{I^2}{A^2}\mathbf{r} + \rho \frac{I^2}{A^2}(\mathbf{e}_3, \mathbf{r})\mathbf{e}_3 + F_1 \mathbf{r} - F_2(\boldsymbol{\xi}_0, \mathbf{r})\boldsymbol{\xi}_0 - \right. \\
&- F_3[(\boldsymbol{\xi}_0, \mathbf{r})\boldsymbol{\xi}_l + (\boldsymbol{\xi}_l, \mathbf{r})\boldsymbol{\xi}_0] - F_4(\boldsymbol{\xi}_l, \mathbf{r})\boldsymbol{\xi}_l + \boldsymbol{\lambda}_{10}]\delta \mathbf{u} dx + \\
&+ \varepsilon(\nabla_{\mathbf{u}} \mathscr{E}[\mathbf{u}_1] + \nabla_{\dot{\mathbf{u}}} \mathscr{D}[\dot{\mathbf{u}}_1], \delta \mathbf{u}) + \int_{\partial V} (\boldsymbol{\lambda}_{20} \times \mathbf{n})\delta \mathbf{u} d\sigma = 0
\end{aligned}
\tag{2.7}
$$

Gauss theorem

$$
\int_V \boldsymbol{\lambda}_{20} \operatorname{rot} \delta \mathbf{u} dx = \int_{\partial V} (\delta \mathbf{u} \times \boldsymbol{\lambda}_{20})\mathbf{n} d\sigma
$$

where $\mathbf{n}$ is the normal to the sphere surface ∂V, was used to obtain the last integral in formula (2.7). Differentiation with respect to time in (2.7) is carried out by virtue of the "unperturbed" system (2.5) and, therefore, in Eq. (2.7)

$$\frac{d}{dt}\left(\rho\frac{l}{A}\mathbf{e}_3\times\mathbf{r}\right)=0$$

Since the work of the elastic and dissipative forces is zero in infinitesimal rotations putting $\delta\mathbf{u} = \delta\boldsymbol{\alpha}\times\mathbf{r}$ in (2.7), we obtain

$$\int_{\partial V}(\boldsymbol{\lambda}_{20}\times\mathbf{n})(\delta\boldsymbol{\alpha}\times\mathbf{r})d\sigma=\frac{8}{3}\pi r_0^3(\boldsymbol{\lambda}_{20},\delta\boldsymbol{\alpha})=0,\quad \forall\delta\boldsymbol{\alpha}\in E^3$$

whence it follows that $\boldsymbol{\lambda}_{20} = 0$. Next, putting $\delta\mathbf{u} = \mathbf{a}$, $\mathbf{a}\in E^3$, we obtain from (2.7) that $\boldsymbol{\lambda}_{10} = 0$.

Hence, the equation for the function $\mathbf{u}_1$ in the first approximation has the form

$$\begin{aligned}&\varepsilon\nabla\mathscr{E}[\mathbf{u}_1+\chi\dot{\mathbf{u}}_1]=\rho\frac{l^2}{A^2}\mathbf{r}-\rho\frac{l^2}{A^2}(\mathbf{r},\mathbf{e}_3)\mathbf{e}_3-F_1\mathbf{r}+F_2(\boldsymbol{\xi}_0,\mathbf{r})\boldsymbol{\xi}_0+\\&+F_3[(\boldsymbol{\xi}_0,\mathbf{r})\boldsymbol{\xi}_1+(\boldsymbol{\xi}_1,\mathbf{r})\boldsymbol{\xi}_0]+F_4(\boldsymbol{\xi}_1,\mathbf{r})\boldsymbol{\xi}_1\end{aligned}\tag{2.8}$$

Here

$$\varepsilon\nabla\mathscr{E}[\mathbf{u}]=-\frac{1}{2(1+\nu)}\left(\frac{1}{1-2\nu}\nabla\operatorname{div}\mathbf{u}+\Delta\mathbf{u}\right)$$

The boundary conditions for the function $\mathbf{u}_1$ consist of the fact that the stresses on the sphere surface are equal to zero

$$\boldsymbol{\sigma}_n=0\tag{2.9}$$

Equation (2.8) can be represented in the form

$$\begin{aligned}&\varepsilon\nabla\mathscr{E}[\mathbf{u}_1+\chi\dot{\mathbf{u}}_1]=\frac{2}{3}\rho\omega_1^2\mathbf{r}+\frac{1}{3}\rho\omega_1^2B_1\mathbf{r}+B_2\mathbf{r}\\&B_1=\operatorname{diag}\{1;1;-2\}\\&B_2\mathbf{r}=-F_1\mathbf{r}+F_2(\boldsymbol{\xi}_0,\mathbf{r})\boldsymbol{\xi}_0+F_3[(\boldsymbol{\xi}_0,\mathbf{r})\boldsymbol{\xi}_1+(\boldsymbol{\xi}_1,\mathbf{r})\boldsymbol{\xi}_0]+F_4(\boldsymbol{\xi}_1,\mathbf{r})\boldsymbol{\xi}_1\\&B_2^T=B_2,\operatorname{tr}B_2=-3F_1+F_2+2F_3\cos\Phi+F_4=0\end{aligned}\tag{2.10}$$

The matrices of the operators B_1 and B_2 are symmetric matrices with a zero trace.

Since Eq. (2.10) is linear, its solution can be represented in the form of the sum of three functions

$$\mathbf{u}_1=\mathbf{u}_{11}+\mathbf{u}_{12}+\mathbf{u}_{13}\tag{2.11}$$

which satisfy the equations

$$\varepsilon\nabla\mathscr{E}[\mathbf{u}_{11}]=\frac{2}{3}\rho\omega_1^2\mathbf{r},\quad\varepsilon\nabla\mathscr{E}[\mathbf{u}_{12}]=\frac{1}{3}\rho\omega_1^2B_1\mathbf{r},\quad\varepsilon\nabla\mathscr{E}[\mathbf{u}_{13}+\chi\dot{\mathbf{u}}_{13}]=B_2\mathbf{r}\tag{2.12}$$

and boundary conditions (2.9).

The solutions of Eqs (2.12) have the form [7]

$$\begin{aligned}&\mathbf{u}_{11}=-\frac{2}{3}\rho\omega_1^2[d_1\mathbf{r}^2+d_2]\mathbf{r}\\&\mathbf{u}_{12}=\frac{1}{3}\rho\omega_1^2\mathbf{u}_{120},\mathbf{u}_{13}\approx\left(1-\chi\frac{d}{dt}\right)\mathbf{u}_{130}\\&\mathbf{u}_{1j0}=a_1(B_{j-1}\mathbf{r},\mathbf{r})\mathbf{r}+a_2\mathbf{r}^2B_{j-1}\mathbf{r}+a_3B_{j-1}\mathbf{r},\quad j=2,3\end{aligned}\tag{2.13}$$

$$d_1 = \frac{(1+\nu)(1-2\nu)}{2(4-3\nu)}, \quad d_2 = -\frac{(3-\nu)(1-2\nu)}{2(4-3\nu)} r_0^2$$

$$a_1 = \frac{1+\nu}{5\nu+7}, \quad a_2 = -\frac{(1+\nu)(2+\nu)}{5\nu+7}, \quad a_3 = \frac{(1+\nu)(2\nu+3)}{5\nu+7} r_0^2$$

The function $\mathbf{u}_{13}$ is represented by the first two terms of the power series in χ under the assumption that $\chi\omega_k l \ll 1$ ($k = 1, 2, 3$). Differentiation with respect to time in the expression for the function $\mathbf{u}_{13}$ is carried out by virtue of the "unperturbed" system (2.5).

The solution found

$$\mathbf{u} = \varepsilon\mathbf{u}_1 = \varepsilon(\mathbf{u}_{11} + \mathbf{u}_{12} + \mathbf{u}_{13})$$

describes the forced oscillations of the viscoelastic sphere. According to the asymptotic method for the separation of motions [1], this solution now has to be substituted into the right-hand sides of Eqs (2.3). In carrying out the above mentioned procedure, triple integrals over the domain V have to be repeatedly calculated. We will now formulate a number of assertions which help to simplify these calculations substantially.

Lemma 2. If

$$\mathbf{u} = -c(d_1\mathbf{r}^2 + d_2)\mathbf{r}$$

then

$$\int_V (\mathbf{r},\mathbf{u})dx = 3cD_1, \quad \int_V (\mathbf{P},\mathbf{r})(\mathbf{Q},\mathbf{u})dx = cD_1(\mathbf{P},\mathbf{Q}); \quad D_1 = \frac{8}{105}\pi r_0^7(1-2\nu)$$

where $\mathbf{R}$, $\mathbf{Q}$ are constants which are independent of $\mathbf{r}$.

Lemma 3. If

$$\mathbf{u} = c[a_1(B\mathbf{r}, \mathbf{r})\mathbf{r} + a_2\mathbf{r}^2 B\mathbf{r} + a_3 B\mathbf{r}]$$

where B is a symmetric matrix with a zero trace, then

$$(\mathbf{P}, \int_V \mathbf{r}\times\mathbf{u}dx) = 0, \quad \int_V (\mathbf{r},\mathbf{u})dx = 0$$

$$\int_V (\mathbf{P},\mathbf{r})(\mathbf{Q},\mathbf{u})dx = cD_2(B\mathbf{P},\mathbf{Q}), \quad D_2 = \frac{4\pi r_0^7(1+\nu)(9\nu+13)}{105(5\nu+7)}$$

and c, $\mathbf{P}$ $\mathbf{Q}$, B are independent of $\mathbf{r}$.

Lemmas 2 and 3 are proved by direct evaluations of the integrals over the sphere.

Substituting the solution $\mathbf{u} = \varepsilon\mathbf{u}_1$ into Eqs (2.3), a closed system of ordinary differential equations in the functions l, Λ, Φ, and β is obtained, containing the small parameters ε, χ, μ, which determines the effect of the forced oscillations of the viscoelastic sphere on its translational – rotational motion. In view of the complexity of these equations, we will represent them in the general form

$$\dot{l} = P_1, \quad \dot{\Lambda} = P_2, \quad \dot{\Phi} = P_3, \quad \dot{\beta} = P_4 \tag{2.14}$$
$$P_k = P_k(l,\Lambda,\Phi,\varepsilon,\chi,\mu), \quad k = 1,2,3,4$$

The right-hand sides of system (2.14) are 2π-periodic with respect to the variable Φ and are independent of the variable β. The fourth equation of (2.14) therefore separates from the equations of this system and can be integrated after the functions l, Λ, Φ from first three equations have been determined. In the following step, we shall consider the system of the first three equations of (2.14) as a system with a small parameter μ with fixed values of ε, and χ. When $\mu = 0$, we obtain

$$\begin{aligned}\dot{I} &= 18\varepsilon\chi D_2\rho^2\omega_2^4(\omega_2-\omega_1)\\ \dot{\Lambda} &= -18\varepsilon\chi D_2\rho^2\omega_2^4(\omega_2-\omega_1)\\ \dot{\Phi} &= \omega_2-\omega_3+6\varepsilon\chi D_2\rho^2\Lambda^{-1}\omega_2^2(\omega_1^2+6\omega_2^2)\end{aligned} \tag{2.15}$$

System (2.15) has the solution

$$I = I(0),\ \Lambda=\Lambda(0),\ \Phi=\omega t+\Phi(0)$$

$$\omega=\omega_2-\omega_3+6\varepsilon\chi D_2\rho^2\omega_2^2(\omega_1^2+6\omega_2^2)/\Lambda(0)$$

and the initial values $I(0)$ and $\Lambda(0)$ are connected by the relation

$$I(0)/A = f^2m^3/(\Lambda(0))^3$$

This motion is taken as the unperturbed motion. It corresponds to a state of gravitational stabilization of the sphere in a circular orbit.

We will seek a solution of the first three equations of (2.14) in the form of expansions [1, 8]

$$I=J_1+\mu N_{11}(J,\psi)+\mu^2N_{12}(J,\psi)+\ldots,\qquad \Lambda=J_2+\mu N_{21}(J,\psi)+\mu^2N_{12}(J,\psi)+\ldots$$

$$\Phi=\psi+\mu M_1(J,\psi)+\mu^2M_2(J,\psi)+\ldots,\qquad J=(J_1,J_2),\ \ \omega_1(J_1)=\omega_2(J_2)$$

where the functions N_{1k}, N_{2k}, M_k $(k = 1, 2, \ldots)$ are 2π-periodic in the variable ψ and have zero mean with respect to this variable.

As functions of time, J_1, J_2, ψ are defined by the differential equations

$$\dot{J}_1=\mu A_{11}(J)+\ldots,\qquad \dot{J}_2=\mu A_{21}(J)+\ldots,\qquad \dot{\psi}=\omega(J)+\mu B_1(J)+\ldots$$

The first approximation equations for the slow variables have the form

$$\begin{aligned}&A_{11}+\frac{\partial N_{11}}{\partial\psi}\omega=-2\varepsilon\chi D_2\Big\{f_{20}\frac{\partial}{\partial\psi}(f_{31}\sin\psi+f_{41}\sin\psi\cos\psi)(\omega_2-\omega_3)-\\ &-f_{20}^2\left(\frac{\partial\omega_2}{\partial J_2}N_{21}-\frac{\partial\omega_1}{\partial J_1}N_{11}+\frac{\partial f_{01}}{\partial J_2}\right)\Big\}\\ &A_{21}+\frac{\partial N_{21}}{\partial\psi}\omega=-\frac{\partial f_{01}}{\partial\psi}+\varepsilon D_2\frac{\partial}{\partial\psi}\Big\{\rho\omega_1^2f_{11}+\frac{2}{3}f_{20}f_{21}+\\ &+\frac{4}{3}f_{20}f_{31}\cos\psi+f_{20}f_{41}\left(\frac{1}{6}+\frac{1}{2}\cos 2\psi\right)\Big\}+\\ &+2\varepsilon\chi D_2\left\{f_{20}\frac{\partial}{\partial\psi}(f_{31}\sin\psi+f_{41}\sin\psi\cos\psi)(\omega_2-\omega_3)-\right.\\ &\left.-f_{20}^2\left(\frac{\partial\omega_2}{\partial J_2}N_{21}-\frac{\partial\omega_1}{\partial J_1}N_{11}+\frac{\partial f_{01}}{\partial J_2}\right)\right\}\end{aligned} \tag{2.16}$$

To determine the functions $A_{11}(J)$ and $A_{21}(J)$, it is necessary to average the right-hand sides of system (2.16) over the angular variable ψ. As a result, we obtain

$$A_{11}=-A_{21}=2\varepsilon\chi D_2f_{20}^2<\frac{\partial f_{01}}{\partial J_2}>\qquad \left(\langle\cdot\rangle=\frac{1}{2\pi}\int_0^{2\pi}(\cdot)d\psi\right)$$

The approximate equations, describing the evolution of the variables I and Λ, are as follows:

$$\dot{J}_1 = -\dot{J}_2 = 36\varepsilon\chi\mu\rho^2 D_2 \omega_2^5 a(q)$$

$$a(q) = \left\langle \frac{1 - q\cos\psi}{(1 - 2q\cos\psi + q^2)^{3/2}} \right\rangle, \qquad \omega_2 = \frac{f^2 m^3}{J_2^3}, \quad q = \frac{b}{R} = \left(\frac{\omega_2}{\omega_3}\right)^{2/3} = \frac{fm^2 b}{J_2^2} \tag{2.17}$$

Taking account of the relations

$$J_2 = m\sqrt{\frac{fb}{q}}, \quad \dot{q} = -\frac{2q}{J_2}\dot{J}_2$$

from (2.17) we obtain the differential equation

$$\dot{q} = \varepsilon\chi\mu n q^9 a(q), \quad n = \frac{72\rho^2 f^2 D_2}{mb^8} > 0 \tag{2.18}$$

for the dimensionless variable q.

Graphs of the function $a(q)$ are shown in Fig. 2. When $q = 1$, the integral, which determines the function $a(q)$, diverges. Since, $q = -bR^{-2}R$, then $R < 0$ when $R > b$ and $R > 0$ when $R < b$. This means that the orbit of the deformed viscoelastic sphere approaches the orbit of a body of mass μ. In this approximation, the angular velocity ω_2 tends to the angular velocity ω_3 and the variable Φ ceases to be a fast variable, which does not allow us to use the scheme of the asymptotic method which has been adopted above.

Graphs of the function $q = q(t)$, obtained by integrating Eq. (2.18) under the assumption that the coefficient $\varepsilon\chi\mu n = 1$, are shown in Fig. 3. If, at the initial instant of time $q(0) > 1$ (inner planets), q decreases and tends to unity while, if $q(0) < 1$. q increases and also tends to unity. As q approaches unity, the rate of evolution increases, since the corresponding integral on the right-hand side of Eq (2.18) diverges when $q = 1$.

If ω_2 and ω_3 are close to one anther, this means that the deformed sphere during its motion in the orbit periodically comes close to the body of mass μ and the problem arises of their mutual capture with the formation of a binary planetary system, similar to the Earth–Moon system. The Sun–Jupiter system can serve as an example of such a situation in the Solar system and the deformable planets are the numerous satellites of Jupiter which have experienced capture by Jupiter during the evolution of their orbital motions around the Sun. The same considerations apply to the Sun–Saturn system where Saturn has numerous satellites.

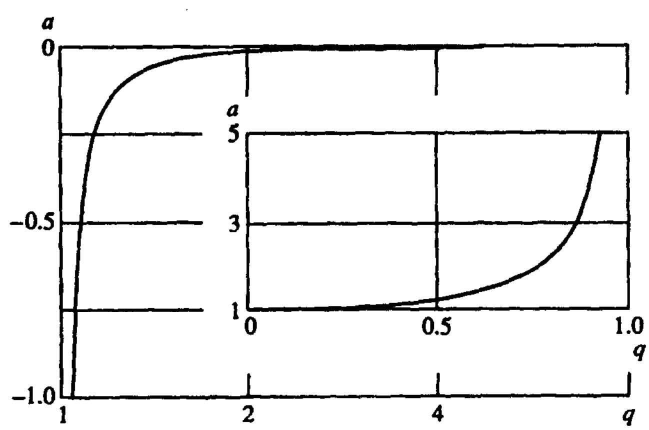

Fig. 2

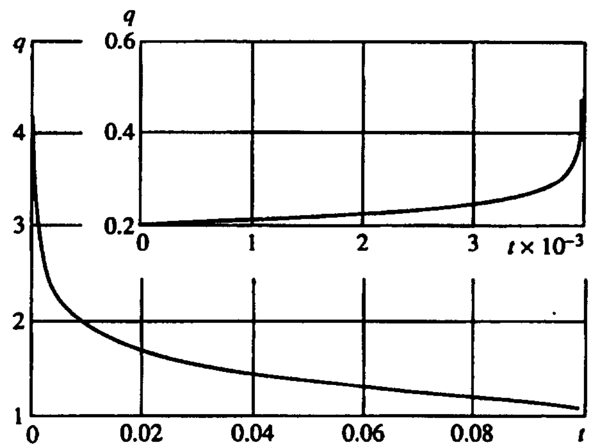

Fig. 3

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